

Ex 1

$$\begin{aligned} \mathcal{L}(X|Y=0) &= \mathcal{U}([0, \theta]) , \quad \theta \in (0,1) \\ \mathcal{L}(X|Y=1) &= \mathcal{U}([0, 1]) \end{aligned} \quad \left. \vphantom{\begin{aligned} \mathcal{L}(X|Y=0) &= \mathcal{U}([0, \theta]) , \quad \theta \in (0,1) \\ \mathcal{L}(X|Y=1) &= \mathcal{U}([0, 1]) \end{aligned}} \right\} \Rightarrow z \in [0,1]$$

$$p = \mathbb{P}(Y=1)$$

$$\eta(x) = \mathbb{P}(Y=1 | X=x).$$

D'après le cours, on a

$$\eta(x) = \frac{p f_+(x)}{p f_+(x) + (1-p) f_-(x)} \quad \text{avec } f_+ \text{ \& } f_- \text{ densités cond. aux classes.}$$

$$= \frac{p \mathbb{1}_{[0,1]}(x)}{p \mathbb{1}_{[0,1]}(x) + (1-p) \frac{\mathbb{1}_{[0,\theta]}(x)}{\theta}}$$

$$\text{Si } \theta < x \leq 1, \quad \eta(x) = p/p = 1$$

Si $x > 1$ ou $x < 0$ (indéfini) impossible pour $x \in [0,1]$

$$\text{Si } 0 \leq x \leq \theta, \quad \eta(x) = \frac{p}{p + (1-p)\frac{1}{\theta}} = \frac{\theta p}{\theta p + 1 - p}$$

Donc, $\eta(x) = \begin{cases} 1 & \text{si } \theta < x \\ \frac{\theta p}{\theta p + 1 - p} & \text{si } 0 \leq x \leq \theta \end{cases}$

si $\theta = 1/2$, $\eta(x) = \begin{cases} 1 & \frac{1}{2} < x \\ \frac{p/2}{p/2 + 1 - p} & 0 \leq x \leq \frac{1}{2} \end{cases}$

Ex 2 ⚠ Dans le cours, $y \in \{-1, 1\}$!

$\mathcal{L}_x = \mathcal{P}_x$, $\mathcal{L}_{y|x=x} = \text{Ber}(\eta(x))$, $\eta(x) = \frac{x}{x + \theta}$,
 $\forall \theta > 0$.

Par définition, $y^*(x) = \mathbb{1}_{\{\eta(x) > 1/2\}}$

Dans notre cas, $\eta(x) > \frac{1}{2} \Leftrightarrow \frac{x}{x + \theta} > \frac{1}{2}$

$\Leftrightarrow \frac{2x}{x + \theta} > 1 \Leftrightarrow x + \theta < 2x$
 $\Leftrightarrow \theta < x$

Donc, $g^*(x) = \mathbb{1}_{\{x > 0\}}$.

Par définition, $L(g^*) = L^* = \mathbb{E}[\min\{\eta(x), 1 - \eta(x)\}]$

$$= \mathbb{E}\left[\mathbb{1}_{\{x \leq 0\}} \eta(x) + \mathbb{1}_{\{x > 0\}} (1 - \eta(x))\right]$$

Si $x \leq 0$, alors $\eta(x) \leq \frac{1}{2}$
 donc $\min\{\eta(x), 1 - \eta(x)\} = \eta(x)$.

On a la densité de la distribution marginale P_X :

$$P_X: \mathbb{R} \rightarrow \mathbb{R}^+$$

$$x \mapsto \frac{\mathbb{1}_{[0, x\theta]}}{x\theta}, \quad \forall x > 1.$$

On peut donc réécrire l'espérance :

$$\begin{aligned} L^* &= \int_{\mathbb{R}} \mathbb{1}_{\{x \leq 0\}} \eta(x) + \mathbb{1}_{\{x > 0\}} (1 - \eta(x)) dP(x) \\ &= \int_{\mathbb{R}} \left[\mathbb{1}_{\{x \leq 0\}} \eta(x) + \mathbb{1}_{\{x > 0\}} (1 - \eta(x)) \right] P_X(x) dx \\ &= \frac{1}{x\theta} \int_0^{x\theta} \mathbb{1}_{\{x \leq 0\}} \eta(x) + \mathbb{1}_{\{x > 0\}} (1 - \eta(x)) dx \end{aligned}$$

$$L^* = \frac{1}{\alpha \theta} \left(\int_0^\theta \gamma(x) dx + \int_\theta^{\alpha\theta} 1 - \gamma(x) dx \right)$$

$$= \frac{1}{\alpha \theta} \left(\int_0^\theta 1 - (1 - \gamma(x)) dx + \int_\theta^{\alpha\theta} 1 - \gamma(x) dx \right)$$

$$= \frac{1}{\alpha \theta} \left(\int_0^\theta 1 - \frac{\theta}{x + \theta} dx + \int_\theta^{\alpha\theta} \frac{\theta}{x + \theta} dx \right)$$

$$= \frac{1}{\alpha \theta} \left(\left[x - \theta \ln(x + \theta) \right]_0^\theta + \left[\theta \ln(x + \theta) \right]_\theta^{\alpha\theta} \right)$$

$$= \frac{1}{\alpha \theta} \left(\theta - \theta \ln(2\theta) + \theta \ln(\theta) + \theta \ln(\alpha\theta + \theta) - \theta \ln(2\theta) \right)$$

$$= \frac{1}{\alpha \theta} \left(\theta - 2\theta \ln(2\theta) + \theta \ln \theta + \theta \ln(\theta(\alpha + 1)) \right)$$

$$= \frac{1}{\alpha \theta} \left(\theta \left(1 - 2 \ln(2\theta) + 2 \ln \theta + \ln(\alpha + 1) \right) \right)$$

$$= \frac{1}{\alpha} \left(1 + 2 \ln \left(\frac{\theta}{2\theta} \right) + \ln(\alpha + 1) \right)$$

$$= \frac{1}{\alpha} \left(1 - 2 \ln 2 + \ln(\alpha + 1) \right)$$

On peut calculer la dérivée :

$$\begin{aligned}\frac{\partial L^*}{\partial \alpha} &= \frac{1}{\alpha} \frac{1}{\alpha+1} - (1 - 2\ln 2 + \ln \alpha + 1) \frac{1}{\alpha^2} \\ &= \frac{1}{\alpha^2} (2\ln 2 - \ln \alpha - 1) + \frac{1}{\alpha} \frac{1}{\alpha+1}\end{aligned}$$

Si on développe, on voit que

$$(*) \quad \frac{\partial L^*}{\partial \alpha} > 0 \Leftrightarrow h > (\alpha+1)e^{\frac{1}{\alpha+1}}$$

Si $\alpha = 1$, $\frac{\partial L^*}{\partial \alpha} > 0$ et si $\alpha = 2$, $\frac{\partial L^*}{\partial \alpha} < 0$. Comme $(\alpha+1)e^{\frac{1}{\alpha+1}}$ est strictement croissant sur $[1, +\infty[$, $\alpha^* \in]1, 2[$.

On peut trouver numériquement $\alpha \approx 1,80$

On peut même se rapporter à $\frac{2^{\alpha+1}}{e} > (\alpha+1)^{\alpha+1}$
(aucune solution analytique). $(*)$

$$\textcircled{x} \quad \frac{\partial L^*}{\partial z} = \frac{1}{x^2} (2 \ln 2 - \ln x_{+1} - 1) + \frac{1}{x} \frac{1}{x_{+1}}$$

$$= \frac{2 \ln 2 - \ln x_{+1} - 1}{x^2} + \frac{1}{x^2 + x}$$

$$= \frac{(2 \ln 2 - \ln x_{+1} - 1)(x^2 + x)}{x^2(x^2 + x)} + \frac{x^2}{x^2(x^2 + x)}$$

$$= \frac{1}{x^4(x^2 + x)} \left(2x^2 \ln 2 - x^2 \ln x_{+1} - x^2 + 2x \ln 2 \right. \\ \left. - x \ln x_{+1} - x + x^2 \right)$$

$$= \frac{x}{x^4(x^2 + x)} (2x \ln 2 - x \ln x_{+1} + 2 \ln 2 - \ln x_{+1} - 1) > 0$$

$$\Rightarrow 2x \ln 2 + 2 \ln 2 > x \ln x_{+1} + \ln x_{+1} + 1$$

$$\Rightarrow (2x + 2) \ln 2 > (x + 1) \ln(x_{+1} + 1) \quad \ln 2 > \frac{1}{2} \ln(x_{+1} + 1) + \frac{1}{2(x_{+1} + 1)}$$

$$\Rightarrow \ln(2^{2x+2}) > \ln((x+1)^{x+1}) + \ln e \quad \ln 2 > \frac{1}{2} \left(\ln(x_{+1}) + \frac{1}{x_{+1}} \right)$$

$$\Rightarrow \ln(2^{2x+2}) > \ln(x_{+1}^{x+1} e)$$

$$\Rightarrow 2^{2x+2} > (x+1)^{x+1} e$$

$$\Rightarrow \frac{2^{2x+2}}{e} > (x+1)^{x+1}$$

$$\ln 2 > \frac{1}{2} \left(\ln(x_{+1}) e^{\frac{1}{x_{+1}}} \right)$$

$$\ln 2 > \ln(x_{+1} e^{\frac{1}{x_{+1}}})$$

$$2 > x_{+1} e^{\frac{1}{x_{+1}}}$$

E_x 3)

$X = (T, U, V)^T$ or T, U, V iid et $T, U, V \sim \mathcal{E}(\lambda)$.

$$Y = \mathbb{1}_{\{T+U+V < \theta\}} \quad \forall \theta > 0.$$

$$L_x = \mathcal{E}(\lambda)^{\otimes 3}$$

1) V n'est pas observé

$$\begin{aligned} \eta(t, u) &= P(T+U+V < \theta \mid T=t, U=u) \\ &= P(V < \theta - t - u \mid T=t, U=u) \end{aligned}$$

On a

$$\begin{aligned} \eta(t, u) &= \mathbb{E} \left[\mathbb{1}_{\{V < \theta - t - u\}} \right] \\ &= \int_0^{\theta - t - u} \underbrace{\mathbb{1}_{\{t+u < \theta\}}}_{\text{condition}} e^{-v} dv \end{aligned}$$

V est ≥ 0 . Si $t+u \geq \theta$, alors V doit être $< \theta - t - u \leq 0$. Impossible.

$$= \mathbb{1}_{\{t+u < \theta\}} \left[-e^{-v} \right]_0^{\theta - t - u} = \mathbb{1}_{\{t+u < \theta\}} (1 - e^{-(\theta - t - u)})$$

On calcule le classifieur de Bayes :

$$g^*(t, u) = \mathbb{1}_{\{q(t, u) > 1/2\}}$$

$$q(t, u) > 1/2 \Leftrightarrow \begin{cases} t+u < \theta \\ e^{-(\theta-t-u)} < 1/2 \end{cases} \Leftrightarrow \begin{cases} t+u < \theta \\ \theta-t-u > \ln 2 \end{cases}$$

$$\Leftrightarrow \theta > t+u + \ln 2$$

$$g^*(t, u) = \mathbb{1}_{\{t+u < \theta - \ln 2\}}$$

On peut calculer l'erreur de Bayes :

$$L^* = \mathbb{E}_{t, u} [\min \{q(t, u); 1 - q(t, u)\}]$$

$$= \mathbb{E}_{t, u} \left[\mathbb{1}_{\{t+u \geq \theta - \ln 2\}} q(t, u) + \mathbb{1}_{\{t+u < \theta - \ln 2\}} (1 - q(t, u)) \right]$$

Calculons la densité de $Z = T + U$:

$$f_Z(z) = \int_{-\infty}^{+\infty} f_T(u) f_U(z-u) du$$

$$\begin{aligned}
 f_z(z) &= \int_{-\infty}^{+\infty} \mathbb{1}_{\mathbb{R}_+}(k) \mathbb{1}_{\mathbb{R}_+}(z-k) e^{-k} e^{-(z-k)} dk \\
 &= \int_0^z e^{-k} e^{-(z-k)} dk \\
 &= \int_0^z e^{-z} dk = e^{-z} \lambda([0, z]) = ze^{-z}.
 \end{aligned}$$

On peut donc ré-écrire l'espérance :

$$L^* = \mathbb{E}_z \left[\mathbb{1}_{\{z \geq \sigma - \ln 2\}} z(z) + \mathbb{1}_{\{z < \sigma - \ln 2\}} (1 - z(z)) \right]$$

avec $z(z) = (1 - e^{-\sigma+z}) \mathbb{1}_{\{z < \sigma\}}$.

$$\begin{aligned}
 L^* &= \int_0^{\sigma - \ln 2} (1 - z(z)) f_z(z) dz + \int_{\sigma - \ln 2}^{\sigma} z(z) f_z(z) dz \\
 &= \int_0^{\sigma - \ln 2} e^{-\sigma+z} z e^{-z} dz + \int_{\sigma - \ln 2}^{\sigma} (1 - e^{-\sigma+z}) z e^{-z} dz \\
 &= \int_0^{\sigma - \ln 2} e^{-\sigma} z dz + \int_{\sigma - \ln 2}^{\sigma} z e^{-z} - z e^{-\sigma} dz \\
 &= e^{-\sigma} \left[\frac{1}{2} z^2 \right]_0^{\sigma - \ln 2} + \left[-z e^{-z} \right]_{\sigma - \ln 2}^{\sigma} + \left[-e^{-z} \right]_{\sigma - \ln 2}^{\sigma} - e^{-\sigma} \left[\frac{1}{2} z^2 \right]_{\sigma - \ln 2}^{\sigma}
 \end{aligned}$$

$$\begin{aligned}
 \int z e^{-z} &= \int_a^b z (e^{-z})' dz \\
 &= [z e^{-z}]_a^b - \int_a^b e^{-z} dz \\
 &= [z e^{-z}]_a^b + [e^{-z}]_a^b
 \end{aligned}$$

$$\begin{aligned}
 L^* &= e^{-\theta} \frac{(\theta - \ln 2)^2}{2} - \theta e^{-\theta} + (\theta - \ln 2) e^{-(\theta - \ln 2)} - e^{-\theta} + e^{-(\theta - \ln 2)} \\
 &\quad - \frac{\theta^2}{2} e^{-\theta} + \frac{(\theta - \ln 2)^2}{2} e^{-\theta}
 \end{aligned}$$

$$= -e^{-\theta} \left(1 + \theta + \frac{\theta^2}{2} - (\theta - \ln 2)^2 \right) + e^{-(\theta - \ln 2)} (1 + (\theta - \ln 2))$$

$$= -e^{-\theta} \left(1 + \theta + \frac{\theta^2}{2} - (\theta - \ln 2)^2 \right) + 2e^{-\theta} (1 + \theta - \ln 2)$$

$$= e^{-\theta} \left(-1 - \theta - \frac{\theta^2}{2} + (\theta - \ln 2)^2 + 2 + 2\theta - 2\ln 2 \right)$$

$$= e^{-\theta} \left(-1 - \theta - \frac{\theta^2}{2} + \theta^2 - 2\theta \ln 2 + \ln^2 2 + 2 + 2\theta - 2\ln 2 \right)$$

$$= e^{-\theta} \left(1 + \theta + \frac{\theta^2}{2} - 2\ln 2 (1 + \theta) + \ln^2 2 \right)$$

Si $\theta = 1$, $L^* \approx 0,0046$.

2) Seulement T est observé.

$$\begin{aligned}\eta(t) &= \mathbb{P}(T+U+V < \theta \mid T=t) \\ &= \mathbb{P}(U+V < \theta-T \mid T=t)\end{aligned}$$

On pose $Z = U+V$.

$$\begin{aligned}\eta(t) &= \mathbb{1}_{\{t < \theta\}} \int_0^{\theta-t} z e^{-z} dz = \mathbb{1}_{\{t < \theta\}} \left([-ze^{-z}]_0^{\theta-t} - [e^{-z}]_0^{\theta-t} \right) \\ &= \mathbb{1}_{\{t < \theta\}} \left((\theta-t) e^{-(\theta-t)} - e^{-(\theta-t)} + 1 \right) \\ &= \mathbb{1}_{\{t < \theta\}} \left(1 + ((\theta-t) - 1) e^{-(\theta-t)} \right) \\ &= \mathbb{1}_{\{t < \theta\}} \left(1 - (\theta-t+1) e^{-(\theta-t)} \right)\end{aligned}$$

$$\begin{aligned}\eta(t) > \frac{1}{2} &\Leftrightarrow \begin{cases} t < \theta \\ 1 - (\theta-t+1) e^{-(\theta-t)} > \frac{1}{2} \end{cases} \\ &\Leftrightarrow \begin{cases} t < \theta \\ (\theta-t+1) e^{-(\theta-t)} < \frac{1}{2} \end{cases}\end{aligned}$$

$$\Leftrightarrow e^{-(\theta-t)} < \frac{1}{2(\theta-t+1)} \quad \wedge \quad t < \theta$$

$$\Leftrightarrow t - \theta < -\ln(2(\theta-t+1)) \quad \wedge \quad t < \theta$$

$$\Leftrightarrow t - \theta < -\ln 2 - \ln(\theta-t+1) \quad \wedge \quad t < \theta$$

$$\Leftrightarrow \theta - t - \ln(\theta-t+1) > \ln 2 \quad \wedge \quad t < \theta$$

$$q(t) > \frac{1}{2} \Leftrightarrow \theta - t - \ln(\theta-t+1) > \ln 2 \quad \wedge \quad t < \theta$$

$$q^*(t) = \mathbb{1}_{\{\theta-t-\ln(\theta-t+1) > \ln 2\}} \mathbb{1}_{\{t < \theta\}}$$

$$L^* = \mathbb{E} \left[\min \{ q(t) ; 1 - q(t) \} \right]$$

$$= \mathbb{E} \left[\mathbb{1}_{\{\theta-t-\ln(\theta-t+1) \leq \ln 2\}}^{(q)} \right]$$

le terme $\mathbb{1}_{\{t < \theta\}}$ est présent dans $q(t)$

$$\mathbb{1}_{\{\theta-t-\ln(\theta-t+1) > \ln 2\}} \mathbb{1}_{\{t < \theta\}} [1 - q(t)]$$

$$= \int_0^\theta \left(\mathbb{1}_{\{\dots\}} 1 - (\theta-t+1)e^{-(\theta-t)} - \mathbb{1}_{\{\dots\}} (\theta-t+1)e^{-(\theta-t)} \right) e^{-t} dt$$

$$\text{Si } \theta = 9$$

$$L^* \approx 0,0059$$

Plus grand que lorsque T et U sont observés.

3) Aucun n'est observé.

On pose $Z = T + U$ et $S = V + Z$. On calcule la densité de S :

$$f_s(s) = \int_0^{+\infty} z e^{-z} e^{-(s-z)} \underbrace{\mathbb{1}_{\mathbb{R}_+}(s-z)}_{\checkmark \in \mathbb{R}_+} dz$$
$$= \int_0^s z e^{-s} dz = e^{-s} \left[\frac{z^2}{2} \right]_0^s = \frac{s^2 e^{-s}}{2}$$

On calcule $\eta(x) = \mathbb{P}(Y=1 | X=x) = \mathbb{P}(S < \theta)$

$$= \int_0^\theta \frac{s^2 e^{-s}}{2} ds$$
$$= \left[-\frac{s^2 + 2s + 2}{2} e^{-s} \right]_0^\theta = 1 - \frac{e^{-\theta}}{2} (\theta^2 + 2\theta + 2)$$

$$\eta(x) \geq \frac{1}{2} \Leftrightarrow e^{-\theta} (\theta^2 + 2\theta + 2) \leq 1$$

$$g^* = \mathbb{1}_{\{e^{-\theta} (\theta^2 + 2\theta + 2) < 1\}}$$

$$L^* = \mathbb{E}[\min \{ \eta(x) ; 1 - \eta(x) \}]$$

$$= \mathbb{E} \left[\mathbb{1}_{\{e^{-\theta} (\theta^2 + 2\theta + 2) \geq 1\}} \eta(x) + \mathbb{1}_{\{e^{-\theta} (\theta^2 + 2\theta + 2) < 1\}} (1 - \eta(x)) \right]$$

$$\text{Si } \theta = 2,$$

$$L^* \approx 0,0062 \quad \text{Encore plus grand que } q^2/2!$$

Ex 4

$$1) f^+(x) = \frac{1}{(2\pi)^{d/2} |\Sigma_+|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_+)^T \Sigma_+^{-1}(x-\mu_+)\right)$$

$$f^-(x) = \frac{1}{(2\pi)^{d/2} |\Sigma_-|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_-)^T \Sigma_-^{-1}(x-\mu_-)\right)$$

$$\gamma = \frac{p f_+}{p f_+ + (1-p) f_-}$$

$$\text{Donc, } \gamma(x) = \frac{\frac{p}{(2\pi)^{d/2} |\Sigma_+|^{1/2}} \exp\left[-\frac{1}{2}(x-\mu_+)^T \Sigma_+^{-1}(x-\mu_+)\right]}{\frac{p}{(2\pi)^{d/2} |\Sigma_+|^{1/2}} \exp\left[-\frac{1}{2}(x-\mu_+)^T \Sigma_+^{-1}(x-\mu_+)\right] + \frac{(1-p)}{(2\pi)^{d/2} |\Sigma_-|^{1/2}} \exp\left[-\frac{1}{2}(x-\mu_-)^T \Sigma_-^{-1}(x-\mu_-)\right]}$$

$$\text{On pose } \pi_{\{+, -\}} = (2\pi)^{d/2} |\Sigma_{\{+, -\}}|^{1/2} \text{ et}$$

$$e_{\{+, -\}} = \exp\left(-\frac{1}{2}((x-\mu_{\{+, -\}})^T \Sigma_{\{+, -\}}^{-1}(x-\mu_{\{+, -\}}))\right)$$

$$\text{Étudions } \frac{p e_+}{\pi_+} + \frac{(1-p) e_-}{\pi_-} :$$

$$\begin{aligned}
 \frac{\pi_+ p e_+ + (1-p)\pi_+ e_-}{\pi_+ \pi_-} &= \frac{\pi_- p e_+ + (1-p)\pi_- e_-}{\pi_+ \pi_-} \\
 &= \frac{p\pi_- e_+ + \pi_+ e_- - p\pi_+ e_-}{\pi_+ \pi_-} \\
 &= \frac{p\pi_- e_+ + (1-p)\pi_+ e_-}{\pi_+ \pi_-}
 \end{aligned}$$

$$\begin{aligned}
 \eta(x) &= \frac{p e_+}{\pi_-} \frac{\pi_+ \pi_-}{p\pi_- e_+ + (1-p)\pi_+ e_-} = \frac{p e_+ \pi_-}{p(\pi_- e_+ + \frac{(1-p)}{p} \pi_+ e_-)} \\
 &= \frac{\pi_- e_+ + \frac{(1-p)}{p} \pi_+ e_-}{\pi_- e_+ + \frac{(1-p)}{p} \pi_+ e_-}
 \end{aligned}$$

$$= \frac{e_+ \pi_-}{e_+ \pi_- \left(1 + \frac{1-p}{p} \frac{\pi_+ e_-}{\pi_- e_+} \right)}$$

$$= \frac{1 + \frac{1-p}{p} \frac{(2\pi)^{d/2} |\Sigma_-|^{1/2} \exp\left(-\frac{1}{2}(x-\mu_-)^T \Sigma_-^{-1}(x-\mu_-)\right)}{(2\pi)^{d/2} |\Sigma_+|^{1/2} \exp\left(-\frac{1}{2}(x-\mu_+)^T \Sigma_+^{-1}(x-\mu_+)\right)}}{1}$$

$$= \frac{1 + \frac{1-p}{p} \left| \frac{\Sigma_+}{\Sigma_-} \right|^{1/2} \exp\left(-\frac{1}{2}(x-\mu_-)^T \Sigma_-^{-1}(x-\mu_-) - (x-\mu_+)^T \Sigma_+^{-1}(x-\mu_+)\right)}{1}$$

$$2) \log \left(\frac{\eta_{\theta}(x)}{1 - \eta_{\theta}(x)} \right) = \theta^T x$$

$$\Rightarrow \frac{\eta_{\theta}(x)}{1 - \eta_{\theta}(x)} = e^{\theta^T x}$$

On peut donc écrire :

$$\begin{aligned} \eta_{\theta}(x) (1 + e^{\theta^T x}) &= \eta_{\theta}(x) + \frac{\eta_{\theta}^2(x)}{1 - \eta_{\theta}(x)} \\ &= \frac{\eta_{\theta}(x) - \eta_{\theta}^2(x) + \eta_{\theta}^2(x)}{1 - \eta_{\theta}(x)} = \frac{\eta_{\theta}(x)}{1 - \eta_{\theta}(x)} = e^{\theta^T x} \end{aligned}$$

On obtient :

$$\eta_{\theta}(x) = \frac{e^{\theta^T x}}{1 + e^{\theta^T x}} = \frac{1}{1 + e^{-\theta^T x}}.$$

Par définition, on sait que $\eta_{\theta}(x) = \frac{p f_+}{f_x}$,
avec $f_x = p f_+ + (1-p) f_-$.

$$\text{On a donc } f_+^{\theta}(x) = \frac{\eta_{\theta}(x) f_x(x)}{p} = \frac{f_x(x)}{p + p e^{-\theta^T x}}.$$

Ex 5)

1) $w \in]0, 1[$.

$$L_w(q) = 2 \mathbb{E}_{x,y} \left[(1-w) \mathbb{1}_{\{y=1\}} \mathbb{1}_{\{q(x)=-1\}} + w \mathbb{1}_{\{y=-1\}} \mathbb{1}_{\{q(x)=1\}} \right]$$

$$= \mathbb{E}_{x,y} \left[(1-w) \underbrace{\mathbb{E}_y [\mathbb{1}_{\{y=1\}} | x]}_{\text{law of total expectation}} \mathbb{1}_{\{q(x)=-1\}} + w \mathbb{E}_y [\mathbb{1}_{\{y=-1\}} | x] \mathbb{1}_{\{q(x)=1\}} \right]$$

$$= \mathbb{E} \left[(1-w) \eta(x) \mathbb{1}_{\{q(x)=-1\}} + w (1-\eta(x)) \mathbb{1}_{\{q(x)=1\}} \right]$$

Donc, $q^*(x) = 1 \Leftrightarrow w(1-\eta(x)) \leq (1-w)\eta(x)$

(q^* minimise L . Donc, si $q^*(x)=1$, $w(1-\eta(x))$ est plus petit que $(1-w)\eta(x)$)

$$\Leftrightarrow w - w\eta(x) \leq \eta(x) - w\eta(x)$$

$$\Leftrightarrow w \leq \eta(x)$$

$$\text{Donc, } q^*(x) = 2 \mathbb{1}_{\{w \leq \eta(x)\}} - 1.$$

Calculons L^* ,

$$\begin{aligned} L_w(q^*) &= 2 \mathbb{E} \left[\mathbb{1}_{\{\eta(X) < w\}} (1-w) \eta(X) + \mathbb{1}_{\{w \leq \eta(X)\}} w (1-\eta(X)) \right] \\ &= 2 \mathbb{E} \left[\left(\mathbb{1}_{\{\eta(X) < w\}} + \mathbb{1}_{\{w \leq \eta(X)\}} \right) \min \left\{ (1-w) \eta(X); w (1-\eta(X)) \right\} \right] \\ &= 2 \mathbb{E} \left[\min \left\{ (1-w) \eta(X); w (1-\eta(X)) \right\} \right] \end{aligned}$$

2) $u \in]0, 1[$

On cherche q tq. $q \in \underset{q}{\operatorname{argmin}} \mathbb{P}(y \neq q(x))$
s.t. $\mathbb{P}(q(x) = 1) = u$.

On peut trouver un q tq $\mathbb{P}(\eta(X) > q) = u$.

Posons $q = q_{1-u}(\eta(X))$ ($\mathbb{P}(\eta(X) \leq q) = 1-u \Leftrightarrow 1 - \mathbb{P}(\eta(X) > q) = 1-u$
 $\Leftrightarrow \mathbb{P}(\eta(X) > q) = u$)

On a $L_u(q) = \mathbb{E} \left[\mathbb{1}_{\{q(X) = -1\}} \eta(X) + \mathbb{1}_{\{q(X) = 1\}} (1-\eta(X)) \right]$

$$= \mathbb{E} \left[(\eta(X) - q) \mathbb{1}_{\{g(X)=1\}} + q \mathbb{1}_{\{g(X)=1\}} + (q - \eta(X)) \mathbb{1}_{\{g(X)=1\}} + (1-q) \mathbb{1}_{\{g(X)=1\}} \right]$$

Or,

$$- \mathbb{E} \left[q \mathbb{1}_{\{g(X)=1\}} \right] = q \mathbb{P}(g(X) \neq 1) = q(1-n)$$

$$- \mathbb{E} \left[(1-q) \mathbb{1}_{\{g(X)=1\}} \right] = (1-q) \mathbb{P}(g(X)=1) = (1-q)n$$

On peut donc écrire

$$L_n(q) = \mathbb{E} \left[(\eta(X) - q) \mathbb{1}_{\{g(X)=1\}} + (q - \eta(X)) \mathbb{1}_{\{g(X)=1\}} \right] + q(1-n) + (1-q)n.$$

Par définition: on, $g^*(X) = 1 \Leftrightarrow q - \eta(X) < \eta(X) - q$

$$\Leftrightarrow -2\eta(X) < -2q$$

$$\Leftrightarrow q < \eta(X)$$

Donc, $g^*(X) = 2 \mathbb{1}_{\{q < \eta(X)\}} - 1$ et, $\mathbb{P}(g^*(X)=1) = \mathbb{P}(\eta(X) > q) = n$.

$$L_n^* = \mathbb{E} \left[\min \{ \eta(X) - q ; q - \eta(X) \} \right] + q(1-n) + (1-q)n.$$

3) θ est une valeur spécifique de rejet.
 $\gamma \in]0, \frac{1}{2}[$.

$$\begin{aligned} L_{\gamma}^{\theta}(y) &= P(Y \neq g(X), g(X) \neq \theta) + \gamma P(g(X) = \theta) \\ &= E[\mathbb{1}_{\{g(X) = -1\}} 2(X) + \mathbb{1}_{\{g(X) = 1\}} (1 - 2(X)) + \gamma P(g(X) = \theta)] \\ &= E[\mathbb{1}_{\{g(X) = -1\}} 2(X) + \mathbb{1}_{\{g(X) = 1\}} (1 - 2(X)) + \gamma \mathbb{1}_{\{g(X) = \theta\}}] \end{aligned}$$

$$S: g^*(X) = 1 \Leftrightarrow 1 - 2(X) < 2(X) \wedge 1 - 2(X) < \gamma$$

$$g^*(X) = -1 \Leftrightarrow 2(X) < 1 - 2(X) \wedge 2(X) < \gamma$$

$$g^*(X) = \theta \Leftrightarrow \gamma < 2(X) \wedge \gamma < 1 - 2(X)$$

Donc

$$g^*(X) = \begin{cases} 1 & \text{s: } 1 - 2(X) < 2(X) \wedge 1 - 2(X) < \gamma \\ -1 & \text{s: } 2(X) < 1 - 2(X) \wedge 2(X) < \gamma \\ \theta & \text{s: } \gamma < 2(X) \wedge \gamma < 1 - 2(X) \end{cases}$$

$$\forall \gamma \in]0, \frac{1}{2}[$$

$$\gamma \leq 1 - \gamma.$$

Donc $2(X) < \gamma$

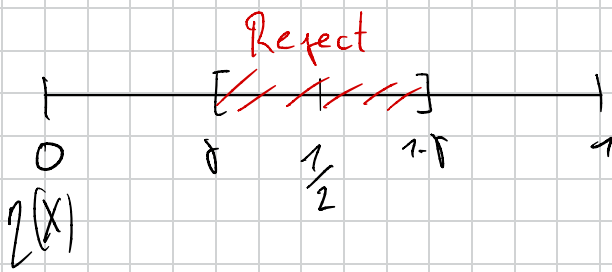
$$\Rightarrow 1 - 2(X) > 1 - \gamma > \gamma > 2(X)$$

et $1 - 2(X) < \gamma$

$$\Rightarrow -2(X) < 1 - \gamma$$

$$\Rightarrow 2(X) > 1 - \gamma > \gamma > 2(X)$$

$$= \begin{cases} 1 & \text{s: } 1 - 2(X) < \gamma \\ -1 & \text{s: } 2(X) < \gamma \\ \theta & \text{s: } \gamma < 2(X) < 1 - \gamma \end{cases}$$



Ex 8)

$$5) A(f) = \mathbb{E}[\max\{0, 2 - \gamma f(x)\}]$$

$$= \mathbb{E}[\max\{0, 1 - f(x)\} \mathbb{1}_{\{\gamma = 1\}} + \max\{0, 1 + f(x)\} \mathbb{1}_{\{\gamma = -1\}}]$$

$$= \mathbb{E}[\max\{0, 1 - f(x)\} \mathbb{E}[\mathbb{1}_{\{\gamma = 1\}} | X] + \max\{0, 1 + f(x)\} \mathbb{E}[\mathbb{1}_{\{\gamma = -1\}} | X]]$$

$$= \mathbb{E}[\max\{0, 1 - f(x)\} \eta(x) + \max\{0, 1 + f(x)\} (1 - \eta(x))]$$

$$= \mathbb{E}[\mathbb{1}_{\{f(x) < 1\}} (1 - f(x)) \eta(x) + \mathbb{1}_{\{f(x) \geq 1\}} (1 + f(x)) (1 - \eta(x))]$$

$$= \mathbb{E}[\mathbb{1}_{\{f(x) < 1\}} (1 - f(x)) \eta(x) > 2\eta(x) \text{ s.t. } f(x) < 1]$$

$$\begin{aligned} \text{s.t. } f(x) \in [-1, 1]. \quad & 2 \min\{\eta(x), 1 - \eta(x)\} \leq \mathbb{1}_{\{-1 < f(x) < 1\}} (1 - f(x)) \eta(x) + (1 + f(x)) (1 - \eta(x)) \leq 2 \max\{\eta(x), 1 - \eta(x)\} \\ & + \underbrace{\mathbb{1}_{\{f(x) > 1\}} (1 + f(x)) (1 - \eta(x))}_{> 2(1 - \eta(x)) \text{ s.t. } f(x) > 1} \end{aligned}$$

$$\begin{aligned} f(x) &= 2 \mathbb{1}_{\{1 - \eta(x) < \eta(x)\}} - 1 \\ &= 2 \mathbb{1}_{\{\frac{1}{2} < \eta(x)\}} \end{aligned}$$

Disposition de cas:

Fixons x .

$$\text{si } q(x) \leq 1 - q(x):$$

$$\text{si } f(x) \leq 1:$$

$$(1 - f(x))q(x) \geq 2q(x)$$

$$\text{si } f(x) \geq 1:$$

$$(1 + f(x))(1 - q(x)) \geq 2(1 - q(x)) \geq 2q(x)$$

$$\text{si } f(x) \in [-1, 1]:$$

$$2q(x) \leq (1 - f(x))q(x) + (1 + f(x))(1 - q(x)) \leq 2(1 - q(x))$$

$2q(x)$ minore donc ces 3 termes. $f(x) = -1$ minimise donc l'intégrale par un x fixé si $q(x) \leq 1 - q(x)$.

De la même manière, si $1 - q(x) \leq q(x)$, $2(1 - q(x))$ minore les 3 termes. $f(x) = 1$ minimise donc l'intégrale par un x fixé si $1 - q(x) \leq q(x)$.

$$\text{Donc, } f(x) = 2 \mathbb{I}_{\{q(x) > \frac{1}{2}\}} - 1.$$